

EQUIVARIANT SYMPLECTIC HOMOLOGY AND MULTIPLE CLOSED REEB ORBITS

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ABSTRACT. We study the existence of multiple closed Reeb orbits on some contact manifolds by means of S^1 -equivariant symplectic homology and the index iteration formula. It is proved that a certain class of contact manifolds which admit displaceable exact contact embeddings, a certain class of prequantization bundles, and Brieskorn spheres have multiple closed Reeb orbits.

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1. INTRODUCTION

After Weinstein's famous conjecture [Wei79], the existence problem of a closed Reeb orbit has been extensively studied. It is a natural question to ask the number of (simple) closed Reeb orbits for contact manifolds which are known to have one closed Reeb orbit. This multiplicity problem has been studied and answered for some classes of contact manifolds as well. We refer to [HWZ98, HWZ03, GHM12] for tight 3-spheres and to [HT09, CGH12] for general contact 3-manifolds. To the authors knowledge, there are few multiplicity result for general higher dimensional contact manifolds but there are a number of theorems [EL80, BLM85, EH87, LZ02, WHL07, Wan11] for pinched or convex hypersurfaces in \mathbb{R}^{2n} .

In the present paper we study the multiplicity problem of closed Reeb orbits for nondegenerate contact manifolds which admit displaceable exact contact embeddings, prequantization bundles, and Brieskorn spheres. Our approach is based on S^1 -equivariant symplectic homology and the index iteration formula. Although we only treat those three cases, we expect that our method can apply for other contact manifolds for which the formulas of S^1 -equivariant symplectic homology (or contact homology) are nice in a sense that will be explained below.

An embedding $i : \Sigma \hookrightarrow W$ of a contact manifold (Σ, ξ) into a symplectic manifold (W, ω) is called an *exact contact embedding* if $i(\Sigma)$ is bounding, $\omega = d\lambda$ for some 1-form λ , and there exists a contact form α on (Σ, ξ) such that $\ker \alpha = \xi$ and $\alpha - \lambda|_\Sigma$ is exact. Throughout this paper we identify $i(\Sigma)$ with Σ . Here by bounding we mean that Σ separates W into two connected components of which one is relatively compact. We denote by W_0 the relatively compact region. This embedding is said to be *displaceable* if there exists a perturbation

$F \in C_c^\infty(S^1 \times W)$ such that the associated Hamiltonian diffeomorphism ϕ_F defined below displaces Σ from itself, i.e. $\phi_F(\Sigma) \cap \Sigma = \emptyset$. A symplectic manifold (W, ω) is called *convex at infinity* if there exists an exhaustion $W = \bigcup_k W_k$ of W by compact sets $W_k \subset W_{k+1}$ with smooth boundaries such that $\lambda|_{\partial W_k}$, $k \in \mathbb{N}$ are contact forms. Simply speaking, we require by convex at infinity that (W, ω) is symplectomorphic to the positive part of the symplectization of a contact manifold at infinity.

The *Reeb vector field* R on (Σ, α) is characterized by $\alpha(R) = 1$ and $i_R d\alpha = 0$. We recall that a closed Reeb orbit is *nondegenerate* if the linearized Poincaré return map associated to the orbit has no eigenvalue equal to 1. A contact form α on (Σ, ξ) is called nondegenerate if every closed Reeb orbit is nondegenerate.

Theorem A. *Suppose that a contact manifold (Σ, ξ) of dimension $2n-1$ admits a displaceable exact contact embedding into (W, ω) which is convex at infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. If either*

- (i) $H_*(W_0, \Sigma; \mathbb{Q}) \neq 0$ for some $* \in 2\mathbb{N} - 1$ or,
- (ii) $H_*(W_0, \Sigma; \mathbb{Q}) = 0$ for all even degree $* \leq 2n - 4$,

then there are at least two closed Reeb orbits contractible in W for any nondegenerate contact form α on (Σ, ξ) .

One may ask if there are more than two closed Reeb orbits when both conditions (i) and (ii) are fulfilled. This question does not seem to be easily answered in general. However the Conley-Zehnder index of closed Reeb orbits on 3-dimensional contact manifolds is special enough to answer this question and the precise statement is given below.

The following list of examples meet the condition (ii) in the theorem.

- (1) (Σ, ξ) is a rational homology sphere;
- (2) (Σ, ξ) is a π_1 -injective fillable 5-manifold;
- (3) (Σ, ξ) is a Weinstein fillable 5-manifold;
- (4) (Σ, ξ) is a subcritical Weinstein fillable 7-manifold.

It is worth pointing out that due to [FSvK12, Lemma 3.4]

$$H_*(\Sigma; \mathbb{Q}) \cong H_{*+1}(W_0, \Sigma; \mathbb{Q}) \oplus H_*(W_0; \mathbb{Q})$$

if Σ is displaceable in W and in particular $H_{*+1}(\Sigma; \mathbb{Q}) = 0$ implies $H_*(W_0, \Sigma; \mathbb{Q}) = 0$

Question. *Does every (nondegenerate) subcritical Weinstein fillable contact manifold has two closed Reeb orbits? More generally, every contact manifold admitting a displaceable exact contact embeddings possesses two closed Reeb orbits?*

We expect that the above question will be answered positively. There is no particular reason that the conditions (i) and (ii) in Theorem A are essential. We include some examples in the appendix which do not meet such conditions but have two closed Reeb orbits.

Aforementioned, it turned out that every 3-dimensional contact manifold has two closed Reeb orbits [CGH12]. Moreover if a nondegenerate contact 3-manifold is not a lens space there are at least three closed Reeb orbits [HT09]. In the following we show that if a contact manifold (Σ, ξ) in Theorem A is of dimension 3, we have not only two closed Reeb orbits but also $b_3(W_0, \Sigma; \mathbb{Q})$ -many closed Reeb orbits.

Corollary A. *Suppose that a 3-dimensional contact manifold (Σ, ξ) admits an exact contact embedding into (W, ω) which is convex at infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. If Σ*

displaceable in (W, ω) , then for any nondegenerate contact form α ,

$$\#\{\text{closed Reeb orbits contractible in } W\} \geq b_3(W_0, \Sigma; \mathbb{Q}) + 2.$$

Moreover, $b_3(W, \Sigma; \mathbb{Q})$ -many simple closed Reeb orbits are of Conley-Zehnder index 2. In particular, if (W, ω) is subcritical Weinstein,

$$\#\{\text{closed Reeb orbits contractible in } W\} \geq b_2(\Sigma; \mathbb{Q}) + 2.$$

Remark 1.1. A closed Reeb orbit γ_0 of Conley-Zehnder index 3 in Corollary 4.4 and $b_3(W, \Sigma; \mathbb{Q})$ -many simple closed Reeb orbits $\{\gamma_1, \dots, \gamma_{b_3(W_0, \Sigma; \mathbb{Q})}\}$ of Conley-Zehnder index 2 in the above corollary have the following nice property. There exist gradient flow lines of the symplectic action functional which connect such closed Reeb orbits with Morse critical points in (W_0, Σ) . These gradient flow lines can be used to obtain finite energy planes. This will be discussed in the forthcoming paper [FK14]. For instance for subcritical Weinstein fillable contact 3-manifolds, using such finite energy planes, we are able to prove that if any closed Reeb orbit is linked with such γ_i , then the linking number is always positive.

As a matter of fact, the proof of Theorem A heavily relies on the facts that the positive part of S^1 -equivariant symplectic homology is periodic, i.e. $\dim SH_*^{S^1, +}(W) = \dim SH_{*+2}^{S^1, +}(W)$ for not small $* \in \mathbb{N}$ and that the positive part of S^1 -equivariant symplectic homology vanishes for low degrees (condition (ii) in Theorem A guarantees this). In other words, we can find more than one closed Reeb orbits if (the positive part of) the S^1 -equivariant symplectic homology of a fillable contact manifold is *nice* in such a sense. A certain class of prequantization bundles and Brieskorn spheres which are treated below have nice S^1 -equivariant symplectic homologies and thus it is able to find more than one closed Reeb orbit.

Let (Q, Ω) be a symplectic manifold with an integral symplectic form Ω , i.e. $[\Omega] \in H^2(Q; \mathbb{Z})$. For each $k \in \mathbb{N}$, there exists a corresponding *prequantization bundle* P over Q with $c_1(P) = k[\Omega]$. Due to [BW58], such a prequantization bundle $(P, \xi := \ker \alpha_{BW})$ is a contact manifold with a connection 1-form α_{BW} . The following theorem proves the existence of two closed Reeb orbits for a certain class of prequantization bundles which naturally arise from the Donaldson's construction, see Remark 4.6.

Theorem B. *Let (P, ξ) be a prequantization bundle over a simply connected integral symplectic manifold (Q, Ω) with $c_1(P) = k[\Omega]$ for some $k \in \mathbb{N}$. Suppose that $[\Omega]$ is primitive in $H^2(Q; \mathbb{Z})$ and $c_1(Q) = c[\Omega]$ for some $|c| > n - 1$ and that (P, ξ) admits an exact contact embedding into (W, ω) with $c_1(W)|_{\pi_2(W)} = 0$ which is π_1 -injective. Then for a nondegenerate contact form α , there are two closed Reeb orbits contractible in W and hence in P .*

More generally, S^1 -orbibundles over symplectic orbifolds provide more examples of contact manifolds. In particular Brieskorn spheres, one of the simplest examples, are our interest for which the positive part of equivariant symplectic homology (contact homology) is already computed in [Ust99]. For $a = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$, we define

$$V_\epsilon(a) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = \epsilon\}$$

which is singular when $\epsilon = 0$. Then a 1-form $\alpha_a = \frac{i}{8} \sum_{j=0}^n a_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$ on $\Sigma_a = V_0(a) \cap S^{2n+1}$ is a contact form. We call $(\Sigma_a, \xi_a := \ker \alpha_a)$ a *Brieskorn manifold*. When n is odd and $a_0 \equiv \pm 1 \pmod{8}$ and $a_1 = \dots = a_n = 2$, Σ_a is diffeomorphic to S^{2n-1} and called a

Brieskorn sphere. As mentioned, Brieskorn manifolds are generalized example of prequantization bundles. Indeed, all Reeb flows of (Σ_a, α_a) are periodic and thus Brieskorn manifolds can be interpreted as principal circle bundles over symplectic orbifolds. Furthermore a Brieskorn manifold is Weinstein fillable and in fact a filling symplectic manifold is V_ϵ with $\epsilon \neq 0$. We refer to [Gei08, Section 7.1] for detailed explanation about Brieskorn manifolds.

Theorem C. *Brieskorn spheres with nondegenerate contact forms have two closed Reeb orbits.*

Except 3-dimensional displaceable case, we only can find two closed Reeb orbits but we do not think that this lower bound is optimal. For instance, it is interesting to ask:

Question. *Can one find more than two closed Reeb orbits on Brieskorn spheres with nonperiodic contact forms?*

1.1. Idea of the proof.

We briefly explain the idea of the proof of Theorem A in the easiest case that a contact manifold is a rational homology sphere. Suppose that a $(2n-1)$ -dimensional contact manifold (Σ, ξ) admits an exact displaceable contact embedding into (W, ω) which is convex at infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. Let α be a corresponding nondegenerate contact form on (Σ, ξ) . If in addition $H_*(\Sigma; \mathbb{Q}) = H_*(S^{2n-1}; \mathbb{Q})$ for all $* \in \mathbb{Z}$, using the vanishing property of $SH^{S^1}(W)$ and the Viterbo long exact sequence, we obtain

$$SH_*^{S^1, +}(W) = \begin{cases} \mathbb{Q} & * = n - 1 + 2k, \ k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

This computation yields there exists at least one simple closed Reeb orbit γ . Assume by contradiction that (Σ, α) has precisely one closed Reeb orbit.

Case 1. $\mu_{CZ}(\gamma) \geq n + 1$.

In this case, $\mu_{CZ}(\gamma^k) + 2 \leq \mu_{CZ}(\gamma^{k+1})$ for all $k \in \mathbb{N}$. Thus $\mu_{CZ}(\gamma) = n + 1$. But there exists $k_0 \in \mathbb{N}$ such that $\mu_{CZ}(\gamma^{k_0}) \neq n - 1 + 2k_0$. This contradicts to the computation (1.1).

Case 2. $\mu_{CZ}(\gamma) < n + 1$.

Since $SH_{\mu_{CZ}(\gamma)}^{S^1, +}(W) = 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu_{CZ}(\gamma^{k_0}) = \mu_{CZ}(\gamma) + 1$ or $\mu_{CZ}(\gamma^{k_0}) = \mu_{CZ}(\gamma) - 1$. But this implies that γ^{k_0} is a bad orbit and does not contribute to the S^1 -equivariant symplectic homology of (W, ω) . Thus we have a contradiction $SH_{\mu_{CZ}(\gamma)}^{S^1, +}(W) \supset \mathbb{Q}\langle \gamma \rangle$.

Remark 1.2. In the first case we can easily find another closed Reeb orbit in the same way using nonequivariant symplectic homology. However in the second case, i.e. $\mu_{CZ}(\gamma) < n + 1$, the Conley-Zehnder index behaves badly under iteration, for instance $\mu_{CZ}(\gamma^{k+1}) < \mu_{CZ}(\gamma^k)$ can occur for some $k \in \mathbb{N}$. Thus it seems difficult to the author to find another closed Reeb orbit using nonequivariant symplectic homology.

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2. S^1 -EQUIVARIANT SYMPLECTIC HOMOLOGY

2.1. Borel type construction.

S^1 -equivariant symplectic homology theory was first introduced in [Vit99]. Recently S^1 -equivariant symplectic homology theory was rigorously studied and written up in [BO09b, BO12b]. In the present paper following [Vit99, BO09a, BO09b] we use the Borel type construction of S^1 -equivariant (Morse-Bott) symplectic homology and refer to [BO12b] for other constructions, their equivalences, and applications.

Let (Σ, ξ) be a contact manifold which admits an exact contact embedding into a symplectic manifold $(W, d\lambda)$ being convex at infinity. Suppose that a corresponding contact form α is nondegenerate. We denote by W_0 the bounded region of $W \setminus \Sigma$. A neighborhood of Σ in W_0 can be trivialized by the Liouville flow as $(\Sigma \times (1 - \epsilon, 1], d(r\alpha))$. The symplectic completion of $(W_0, d\lambda)$ is defined by

$$\widehat{W} = W_0 \cup_{\partial W_0} \Sigma \times [1, \infty), \quad \widehat{\omega} = \begin{cases} d\lambda & \text{on } W_0, \\ d(r\alpha) & \text{on } \Sigma \times [1, \infty). \end{cases}$$

We denote by $\widehat{\lambda}$ a primitive 1-form of $\widehat{\omega}$ which is λ on W_0 and $r\alpha$ on $\Sigma \times [1, \infty)$.

We choose an almost complex structure J on W_0 which is compatible with ω and preserves the contact hyperplane field $\ker \alpha \subset T\Sigma$. We extend this on \widehat{W} so that J is invariant under the \mathbb{R}_+ -action and $Jr\partial_r = R$ and $JR = -r\partial_r$. Such a J will be called admissible. Here R is the Reeb vector field associated to α and we denote by φ_R^t the flow of R . The *Hamiltonian vector field* X_H associated to a Hamiltonian function $H \in C^\infty(S^1 \times \widehat{W})$ is defined by $i_{X_H}\widehat{\omega} = dH$.

Since we have assumed that (Σ, α) is nondegenerate, periods of closed Reeb orbits on (Σ, α) forms a discrete subset $\text{Spec}(\Sigma, \alpha)$ in $\mathbb{R}_+ := (0, \infty)$. We define a *family of S^1 -invariant admissible Hamiltonians* $K_\tau \in C^\infty(\widehat{W} \times S^{2N+1})$, $\tau \in \mathbb{R}_+ \setminus \text{Spec}(\Sigma, \alpha)$ to have the following properties:

- (i) $K_\tau(x, z) = H_\tau(x) + h(z)$ for $(x, z) \in \widehat{W} \times S^{2N+1}$;
- (ii) $h \in C^\infty(S^{2N+1})$ is invariant under the S^1 -action on S^{2N+1} ;
- (iii) On W , $H_\tau < 0$ and is a C^2 -small Morse function;
- (iv) On $\Sigma \times [1, \infty)$, $H_\tau(x) = h_\tau(r)$ for some strictly increasing function $h_\tau : [1, \infty) \rightarrow \mathbb{R}_+$ satisfying $h_\tau''(r) > 0$ on $(1, r_0)$ for some $r_0 > 0$;
- (v) $h_\tau(r) = \tau r - \tau$ on $\Sigma \times (r_0, \infty)$.

With a family of admissible Hamiltonians $K_\tau \in C^\infty(\widehat{W} \times S^{2N+1})$, we define a family of action functionals $\mathcal{A}_{K_\tau}^N : \mathcal{L}_{\widehat{W}} \times S^{2N+1} \rightarrow \mathbb{R}$, where $\mathcal{L}_{\widehat{W}}$ denotes the free loop space of \widehat{W} , by

$$\mathcal{A}_{K_\tau}^N(v, z) := - \int_{S^1} v^* \widehat{\lambda} - \int_{S^1} K_\tau(v, z) dt.$$

We note that this action functional is S^1 -invariant with respect to the diagonal action of S^1 on $\mathcal{L}_W \times S^{2N+1}$,

$$\theta \cdot (v(t), z) := (v(t - \theta), e^{2\pi i \theta} z), \quad t \in S^1, z \in S^{2N+1} \subset \mathbb{C}^{N+1}.$$

That is $\mathcal{A}_{K_\tau}^N(\theta v, \theta \lambda) = \mathcal{A}_{K_\tau}^N(v, \lambda)$, and thus the critical points set $\text{Crit} \mathcal{A}_{K_\tau}^N$ is S^1 -invariant as well. Here $(v, z) \in \text{Crit} \mathcal{A}_{K_\tau}^N$ if and only if

$$\begin{cases} \partial_t v - X_{H_\tau}(v) = 0, \\ d_z f(z) = 0. \end{cases}$$

We denote an S^1 -family of critical points containing (v, z) by

$$S_{(v,z)} := \{\theta \cdot (v, z) \mid (v, z) \in \text{Crit} \mathcal{A}_{K_\tau}^N, \theta \in S^1\}.$$

There are two types of critical points of $\mathcal{A}_{K_\tau}^N$:

- 1) (v, z) where $z \in \text{Crit} f$ and where $v \equiv x \in W_0$ is a critical point of the Morse function $H_\tau|_W$;
- 2) (v, z) where $z \in \text{Crit} f$ and where $v \in \mathcal{L}_{\widehat{W}}$ lying on levels $\Sigma \times \{r\}$, $r \in (1, r_0)$ is a solution of

$$\partial_t v = -h'_\tau(\pi \circ v)R(v). \quad (2.1)$$

Here $\pi : \Sigma \times [1, \infty) \rightarrow [1, \infty)$ is the projection to the second factor.

The second type solutions correspond to closed Reeb orbits with period $h'_\tau(\pi \circ v) \in (0, \tau)$. They are transversally nondegenerate (see [BO09a, Lemma 3.3]), i.e.

$$\ker[d\varphi_R^{-h'_\tau(\pi \circ v)}(v(0)) - \mathbb{1}_{T_{v(0)}\widehat{W}}] = \langle \partial_t v(0) \rangle.$$

Suppose that $c_1(W)|_{\pi_2(W)} = 0$. Then we are able to associate the parametrized index function μ to critical points of $\mathcal{A}_{K_\tau}^N$

$$\mu : \text{Crit} \mathcal{A}_{K_\tau}^N \longrightarrow \mathbb{Z},$$

and due to the splitting property

$$\mu(v, z) = \mu_{CZ}(v) + \text{ind}_f(z)$$

where μ_{CZ} and ind_f stand for the Conley-Zehnder index and the Morse index respectively, see [BO12a]. In particular if $(v, z) \in \text{Crit} \mathcal{A}_{K_\tau}^N$ is of the first type, i.e. $v = x \in W_0$,

$$\mu(v, z) = \mu_{CZ}(v) + \text{ind}_f(z) = \text{ind}_{-H_\tau|_W}(x) - \frac{\dim W}{2} + \text{ind}_f(z).$$

We also define a *family of S^1 -invariant admissible almost complex structures* $J = (J_\lambda^t)$, $\lambda \in S^{2N+1}$, $t \in S^1$ such that J_λ^t is an admissible almost complex structure on \widehat{W} and is S^1 -invariant, i.e. $J_\lambda^t = J_{\theta\lambda}^{t-\theta}$ for $\theta \in S^1$. Together with a Riemannian metric g on S^{2N+1} invariant under the S^1 -action, a metric on $\mathcal{L}_{\widehat{W}} \times S^{2N+1}$ is defined by

$$m_{(v,z)}((\xi_1, \zeta_1), (\xi_2, \zeta_2)) := \int_{S^1} \omega(\xi_1, J_{\xi_2}^t) dt + g(\zeta_1, \zeta_2), \quad (\xi_i, \zeta_i) \in T_v \mathcal{L}_{\widehat{W}} \times T_z S^{2N+1}.$$

A gradient flow line $(u, y) : C^\infty(\mathbb{R} \times S^1, \widehat{W}) \times C^\infty(\mathbb{R}, S^{2N+1})$ of $\mathcal{A}_{K_\tau}^N$ with respect to the metric m is a solution of

$$\begin{cases} \partial_s u + J_{y(s)}^t (\partial_t u - X_{H_\tau}(u)) = 0, \\ \partial_s y - \nabla_g f(y) = 0. \end{cases} \quad (2.2)$$

We denote by $\widehat{\mathcal{M}}(S_{(v_-, z_-)}, S_{(v_+, z_+)})$ the moduli space of gradient flow lines from (v_-, z_-) to (v_+, z_+) , i.e.

$$\begin{aligned} \widehat{\mathcal{M}}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) &= \widehat{\mathcal{M}}(S_{(v_-, z_-)}, S_{(v_+, z_+)}; K_\tau, J, g) \\ &:= \left\{ (u, y) \mid (u, y) \text{ solves (2.2) and } \lim_{s \rightarrow \pm\infty} (u, y)(s) \in S_{(v_\pm, z_\pm)} \right\}. \end{aligned}$$

We divide out the \mathbb{R} -action on $\widehat{\mathcal{M}}(S_{(v_-, z_-)}, S_{(v_+, z_+)})$ defined by shifting gradient flow lines in the s -variable. Then we have the moduli space of unparametrized gradient flow lines denoted by

$$\mathcal{M}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) := \widehat{\mathcal{M}}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) / \mathbb{R}$$

We note that solutions of (2.2) is S^1 -equivariant, that is if (u, y) solves (2.2), then so does $\theta \cdot (u, y)$, and this S^1 -action freely acts on the moduli space $\mathcal{M}(S_{(v_-, z_-)}, S_{(v_+, z_+)})$. We denote the quotient by

$$\mathcal{M}_{S^1}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) := \mathcal{M}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) / S^1.$$

It turned out that this moduli space is a smooth manifold of dimension

$$\dim \mathcal{M}_{S^1}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) = \mu(v_-, z_-) - \mu(v_+, z_+) - 1$$

for a generic J . For the detailed transversality analysis we refer to [BO10]. We define the S^1 -equivariant chain group $SC_*^{S^1, N}(K_\tau)$ by the \mathbb{Q} -vector space generated by S^1 -families of critical points of $\mathcal{A}_{K_\tau}^N$ of μ -index $*$ $\in \mathbb{Z}$.

$$SC_*^{S^1, N}(K_\tau) = \bigoplus_{\substack{S(v, z) \in \text{Crit} \mathcal{A}_{K_\tau}^N \\ \mu(v, z) = *}} \mathbb{Q} \langle S_{(v, z)} \rangle.$$

The boundary operator $\partial^{S^1} : SC_*^{S^1, N}(K_\tau) \rightarrow SC_{*-1}^{S^1, N}(K_\tau)$ is defined by

$$\partial^{S^1}(S_{(v_-, z_-)}) = \sum_{\substack{S(v_+, z_+) \in \text{Crit} \mathcal{A}_{K_\tau}^N \\ \mu(v_-, z_-) - \mu(v_+, z_+) = 1}} \# \mathcal{M}_{S^1}(S_{(v_-, z_-)}, S_{(v_+, z_+)}) S_{(v_+, z_+)}$$

where by $\#$ we mean a signed (via the coherent orientations) count of the number of the finite set $\mathcal{M}_{S^1}(S_{(v_-, z_-)}, S_{(v_+, z_+)})$. Then $\partial^{S^1} \circ \partial^{S^1} = 0$ and thus we are able to define

$$HF_*^{S^1, N}(K_\tau) = H_*(SC_*^{S^1, N}(K_\tau), \partial^{S^1}).$$

Taking direct limits, the S^1 -equivariant symplectic homology of (W, ω) is defined by

$$SH_*^{S^1}(W) := \lim_{N \rightarrow \infty} \lim_{\tau \rightarrow \infty} HF_*^{S^1, N}(K_\tau)$$

Here the direct limit of N is taken with respect to the embedding $S^{2N-1} \hookrightarrow S^{2N+1}$. The resulting homology depends only on (W_0, ω) . In order to define the negative/positive part of S^1 -equivariant symplectic homology, we consider

$$SC_*^{S^1, -, N}(K_\tau) = \bigoplus_{\substack{S(v, z) \in \text{Crit} \mathcal{A}_{K_\tau}^N \\ \mathcal{A}_{K_\tau}^N(v, z) < \epsilon}} \mathbb{Q} \langle S_{(v, z)} \rangle, \quad SC_*^{S^1, +, N}(K_\tau) = SC_*^{S^1, N}(K_\tau) / SC_*^{S^1, -, N}(K_\tau)$$

where $\epsilon < \min \text{Spec}(\Sigma, \alpha)$. That is, $SC_*^{S^1, -, N}$ resp. $SC_*^{S^1, +, N}$ is generated by type 1) resp. type 2) critical points of $\mathcal{A}_{K_\tau}^N$. Since the action values decrease along gradient flow lines, there

exist associated boundary operators $\partial_{\pm}^{S^1}$ induced by ∂^{S^1} , and hence we are able to define $SH_*^{S^1, \pm}(W)$ the negative/positive part of S^1 -equivariant symplectic homology of (W, ω) .

2.2. Morse-Bott spectral sequence.

This subsection is devoted to observe that bad orbits do not contribute to S^1 -equivariant symplectic homology which is certainly expected to be true in S^1 -equivariant theory. To see this feature we use a Morse-Bott spectral sequence. We refer to [Fuk96] for detailed explanation about the Morse-Bott spectral sequence. We should mention that this approach was used by [FSvK12] to study the non-existence of a displaceable exact contact embedding of Brieskorn manifolds.

There is a Morse-Bott spectral sequence which converges to $SH_*^{S^1, +}(W)$ whose first page is given by

$$E_{j,i}^1 = \bigoplus_{\substack{\gamma \in \mathcal{P} \\ \mu_{CZ}(\gamma)=i}} H_j(\gamma \times_{S^1} ES^1; \mathcal{O}_{\gamma})$$

where \mathcal{O}_{γ} is a orientation rational bundle of γ . We note that if γ is a k -fold cover of a simple Reeb orbit, $\gamma \times_{S^1} ES^1$ is the infinite dimensional lens space $B\mathbb{Z}_k$. We recall that parities of Conley-Zehnder indices of all even/odd multiple covers of a simple closed Reeb orbits are the same, i.e.

$$\mu_{CZ}(\gamma^{2k}) \equiv \mu_{CZ}(\gamma^{2\ell}), \quad \mu_{CZ}(\gamma^{2k+1}) \equiv \mu_{CZ}(\gamma^{2\ell+1}) \pmod{2}, \quad k, \ell \in \mathbb{N}.$$

See [Vit89, Ust99] for instance. A closed Reeb orbit γ is called *bad* if $\gamma = \gamma_0^k$ for a simple Reeb orbit γ_0 and some $k \in \mathbb{N}$ (if fact, $k \in 2\mathbb{N}$) and the parity of $\mu_{CZ}(\gamma)$ disagrees with the parity of $\mu_{CZ}(\gamma_0)$. A closed Reeb orbit which is not bad is called *good*. If γ is a good orbit, the twist bundle \mathcal{O}_{γ} is trivial and $H_j(B\mathbb{Z}_k; \mathbb{Q})$ vanishes except degree zero. If γ is a bad orbit, \mathcal{O}_{γ} is the orientation bundle of $B\mathbb{Z}_k$ and $H_j(B\mathbb{Z}_k; \mathcal{O}_{\gamma})$ vanishes for every degree, see [Vit89]. Therefore only good closed Reeb orbits contribute to the first page of the Morse-Bott spectral sequence and thus to the positive part of S^1 -equivariant symplectic homology as well.

2.3. Resonance identity.

Following [vK05] we define the mean Euler characteristic by

$$\chi_m(W) := \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{\ell=-N}^N (-1)^{\ell} \dim SH_{\ell}^{S^1, +}(W)$$

From the observation of the previous subsection we know the first page of the Morse-Bott spectral sequence converging to $SH^{S^1, +}(W)$ is given by

$$E_{j,i}^1 = \bigoplus_{\substack{\gamma \in \mathfrak{G} \\ \mu_{CZ}(\gamma)=i}} \mathbb{Q}$$

where \mathfrak{G} is the set of good closed Reeb orbits contractible in W . Since the mean Euler characteristic of E^1 is the same as that of $SH^{S^1, +}(W)$, we have

$$\chi_m(W) = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{\gamma \in \mathfrak{G}_N} (-1)^{\mu_{CZ}(\gamma)}$$

where \mathfrak{G}_N is the set of good closed Reeb orbits of Conley-Zehnder indices in $[-N, N]$. Let $\Delta(\gamma)$ be the mean Conley-Zehnder index of γ which will be explained in the next section. From $|\mu_{CZ}(\gamma^k) - k\Delta(\gamma)| < n - 1$, see [SZ92], we have

$$k\Delta(\gamma) - (n - 1) < \mu_{CZ}(\gamma^k) < k\Delta(\gamma) + (n - 1), \quad \Delta(\gamma) = \lim_{k \rightarrow \infty} \frac{\mu_{CZ}(\gamma^k)}{k}.$$

Thus there exist constants $C_1, C_2 \in [-(n - 1), n - 1]$ such that $\mu_{CZ}(\gamma^k) \in [-N, N]$ if and only if

$$\max \left\{ 0, \frac{-N + C_1}{\Delta(\gamma)} \right\} < k < \frac{N + C_2}{\Delta(\gamma)}. \quad (2.3)$$

We abbreviate by \mathfrak{G}_s the set of simple closed Reeb orbits contractible in W whose multiple covers are all good and by \mathfrak{B}_s the set of simple closed Reeb orbits contractible in W whose even multiple covers are bad. Then (2.3) implies the following proposition. This idea is basically identical to [GK10].

Proposition 2.1. *Let (W, ω) be as above. Then we have*

$$\chi_m(W) = \sum_{\gamma_g \in \mathfrak{G}_s} \frac{(-1)^{\mu_{CZ}(\gamma_g)}}{\Delta(\gamma_g)} + \sum_{\gamma_b \in \mathfrak{B}_s} \frac{(-1)^{\mu_{CZ}(\gamma_b)}}{2\Delta(\gamma_b)}. \quad (2.4)$$

3. INDEX ITERATION FORMULA

In the present section, we first recall the Conley-Zehnder index of a closed Reeb orbit and then briefly explain how the Conley-Zehnder index varies under iteration. For detailed explanation we refer to Long's book [Lon02], see also [CZ84, SZ92, Sal99, Gut12]. For the sake of compatibility, we will adopt the notation and terminology of [Lon02].

Let $\mathrm{Sp}(2n)$ be the space of $2n \times 2n$ symplectic matrices and $\mathrm{Sp}(2n)^*$ be a subset which consists of nondegenerate elements, i.e.

$$\mathrm{Sp}(2n)^* := \{M \in \mathrm{Sp}(2n) \mid \det(M - \mathbb{1}_{2n}) \neq 0\}.$$

We observe that $\mathrm{Sp}(2n)^* = \mathrm{Sp}(2n)^+ \cup \mathrm{Sp}(2n)^-$ where

$$\mathrm{Sp}(2n)^\pm := \{M \in \mathrm{Sp}(2n) \mid \pm \det(M - \mathbb{1}_{2n}) > 0\}.$$

An element $M \in \mathrm{Sp}(2n)$ is called *elliptic* if the spectrum $\sigma(M)$ is contained in the unit circle $U := \{z \in \mathbb{C} \mid |z| = 1\}$. Since we are interested in the nondegenerate case, i.e. $M \in \mathrm{Sp}(2n)^*$, $\sigma(M) \subset U \setminus \{1\}$. The *elliptic height* of M is defined by the total algebraic multiplicity of all eigenvalues of M in U and denoted by $e(M)$. On the other hand if $\sigma(M) \cap U = \emptyset$, i.e. $e(M) = 0$, M is called *hyperbolic*.

We abbreviate

$$\mathcal{P}(2n, \tau)^* := \{\Psi : [0, \tau] \rightarrow \mathrm{Sp}(2n)^* \mid \Psi(0) = \mathbb{1}_{2n}\}.$$

For $\Psi \in \mathcal{P}(2n, \tau)^*$, we join $\Psi(\tau) \in \mathrm{Sp}(2n)^\pm$ to

$$-\mathbb{1}_{2n} \quad \text{or} \quad \mathrm{diag}(2, 1/2, -1, \dots, -1)$$

by a path $\psi : [0, 1] \rightarrow \mathrm{Sp}(2n)^*$. We recall that there exists a continuous map

$$\rho : \mathrm{Sp}(2n) \longrightarrow S^1$$

which is uniquely characterized by the naturality, the determinant, and the normalization properties. For any path $r : [0, c] \rightarrow \mathrm{Sp}(2n)$, we choose a function $\alpha_r : [0, c] \rightarrow \mathbb{R}$ such that $\rho(r(t)) = e^{i\alpha_r(t)}$. Then the Maslov-type index for a path $\Psi \in \mathcal{P}(2n, \tau)^*$ is defined by

$$\mu(\Psi) := \frac{\alpha_\Psi(\tau) - \alpha_\Psi(0)}{\pi} + \frac{\alpha_\psi(1) - \alpha_\psi(0)}{\pi} \in \mathbb{Z}.$$

In particular, we denote

$$\Delta(\Psi) := \frac{\alpha_\Psi(\tau) - \alpha_\Psi(0)}{\pi} \in \mathbb{R}.$$

and call the *mean index* of γ . We remark that since $\mathrm{Sp}(2n)^*$ is simply connected, both μ and Δ are independent of the choice of a path ψ .

Now we associate this Maslov-type index to each closed Reeb orbit contractible in a symplectic filling. Let γ be an τ -period closed Reeb orbit on (Σ, α, ξ) contractible in (W, ω) . We take a filling disk $\bar{\gamma} : D^2 \rightarrow W$ such that $\bar{\gamma}|_{\partial D^2} = \gamma$. Then a symplectic trivialization $\Phi : \bar{\gamma}^* \xi \rightarrow D^2 \times \mathbb{R}^{2n-2}$ and the linearized flow $T\phi_R^t(\gamma(0))|_\xi$ along γ induces a path of symplectic matrices

$$\Psi_\gamma(t) := \Phi(\gamma(t)) \circ T\phi_R^t(\gamma(0))|_\xi \circ \Phi^{-1}(\gamma(0)) : [0, \tau] \rightarrow \mathrm{Sp}(2n-2).$$

If γ is nondegenerate, $\Psi_\gamma \in \mathcal{P}(2n-2, \tau)^*$ and we are able to define the *Conley-Zehnder index* of γ by

$$\mu_{CZ}(\gamma) := \mu(\Psi_\gamma).$$

The mean Conley-Zehnder index $\Delta(\gamma)$ is also defined in a trivial way. In order to prove our main results we need to study the Conley-Zehnder indices of γ^k , $k \in \mathbb{N}$ where

$$\gamma^k : [0, k\tau] \rightarrow \Sigma, \quad \gamma^k(t) := \gamma(t - j\tau) \text{ for } t \in [j\tau, (j+1)\tau], \quad 1 \leq j \leq k-1.$$

We define the k -th iteration $\Psi^k \in \mathcal{P}(2n-2, k\tau)^*$ of $\Psi \in \mathcal{P}(2n-2, \tau)^*$ by

$$\Psi^k(t) := \Psi(t - j\tau)\Psi(\tau)^j, \quad t \in [j\tau, (j+1)\tau], \quad 1 \leq j \leq k-1$$

so that $\Psi_{\gamma^k} = \Psi_\gamma^k$ and $\mu_{CZ}(\gamma^k) = \mu(\Psi_\gamma^k)$.

Let M_1 resp. M_2 be $2i \times 2i$ resp. $2j \times 2j$ matrix of the square block form as below.

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

The \diamond -product of M_1 and M_2 is a $2(i+j) \times 2(i+j)$ matrix defined by

$$M_1 \diamond M_2 := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

The following symplectic matrices are called *basic normal forms*.

- $D(\pm 2) = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 1/2 \end{pmatrix},$
- $N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0,$
- $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$
- $N_2(\theta, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}$ for $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), b_2 \neq b_3.$

We note that $D(\pm 2)$ are basic normal forms for eigenvalues outside U and N_1, R , and N_2 are basic normal forms for eigenvalues in U . Therefore $e(D) = 0$, $e(N_1) = e(R) = 2$, and $e(N_2) = 4$.

The *homotopy set* $\Omega(M)$ of $M \in \text{Sp}(2n)$ is defined by

$$\Omega(M) = \{M' \in \text{Sp}(2n) \mid \sigma(M') \cap U = \sigma(M) \cap U, \nu_\lambda(M') = \nu_\lambda(M) \text{ for all } \lambda \in \sigma(M) \cap U\}$$

where

$$\nu_\lambda(M) := \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - \lambda \mathbb{1}_{2n}).$$

We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ containing M .

Theorem 3.1. *For $M \in \text{Sp}(2n)$, there exists a path $h : [0, 1] \rightarrow \Omega^0(M)$ such that*

$$h(0) = M \quad \text{and} \quad h(1) = M_1 \diamond \cdots \diamond M_k \diamond M_0$$

where M_i 's, $i \in \{1, \dots, k\}$ are basic normal forms for eigenvalues in U and M_0 is either $D(2)^{\diamond \ell}$ or $D(-2) \diamond D(2)^{\diamond(\ell-1)}$ for some $\ell \in \mathbb{N}$.

PROOF. The proof can be found in [Lon02, Theorem 1.8.10 & Corollary 2.3.8] □

Since we are interested in the nondegenerate case, i.e. $M \in \text{Sp}(2n)$ with $\nu_1(M^k) = 0$ for all $k \in \mathbb{N}$, we can exclude the basic normal form $N_1(\lambda, b)$ since

$$\nu_1(N_1(1, b)^k) \geq 1, \quad \text{for some } k \in \mathbb{N}.$$

Moreover,

$$\nu_1(R(\theta)^k) = 2 - 2\varphi\left(\frac{k\theta}{2\pi}\right), \quad \nu_1(N_2(\theta, B)^k) = 2 - 2\varphi\left(\frac{k\theta}{2\pi}\right), \quad k \in \mathbb{N},$$

where $\varphi(a) = 0$ if $a \in \mathbb{Z}$ and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$. Thus $\theta/2\pi$ should be irrational due to the nondegeneracy condition. Therefore in the case at hand, the endpoint of the path $h : [0, 1] \rightarrow \Omega^0(M)$ in Theorem 3.1 is simply

$$h(1) = R(\theta_1) \diamond \cdots \diamond R(\theta_p) \diamond N_2(\theta_{p+1}, B_1) \diamond \cdots \diamond N_2(\theta_{p+q}, B_q) \diamond M_0 \quad (3.1)$$

for $\theta_i/2\pi \in (0, 1) \setminus \mathbb{Q}$, $i \in \{1, \dots, p+q\}$. Now we are ready to state the following theorem due to [Lon00] which will plays a crucial role.

Theorem 3.2. *Let $\Psi \in \mathcal{P}(2n, \tau)$ with $\Psi(\tau)^k \in \text{Sp}(2n)^*$ for all $k \in \mathbb{N}$, i.e. $\Psi^k \in \mathcal{P}(2n, k\tau)^*$, and $h : [0, 1] \rightarrow \Omega^0(\Psi(\tau))$ such that $h(0) = \Psi(\tau)$ and $h(1)$ is as (3.1). Then the Maslov index of Ψ^k is*

$$\mu(\Psi^k) = \sum_{1 \leq i \leq p} \left(k(P_i - 1) + 2 \left\lfloor \frac{k\theta_i}{2\pi} \right\rfloor + 1 \right) + \sum_{1 \leq j \leq q} kW_j + \sum_{1 \leq o \leq \ell} kQ_o$$

where P_i 's are odd integers and W_j 's, Q_o 's are integers and they satisfy

$$\sum_{i,j,o} P_i + W_j + Q_o = \mu(\Psi).$$

Here, $[a] \in \mathbb{Z}$ is the biggest integer number smaller than or equal to $a \in \mathbb{R}$.

PROOF. The proof can be found in [Lon00] or [Lon02, Chapter 8]. \square

Example 3.3. Let γ be a simple closed Reeb orbit on a contact manifold of dimension 3 and suppose that all γ^k 's are nondegenerate. If γ is elliptic, $\mu_{CZ}(\gamma) \in 2\mathbb{Z} + 1$ and

$$\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 1) + 2[k\theta] + 1, \quad \theta \in (0, 1) \setminus \mathbb{Q}.$$

If γ is hyperbolic,

$$\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).$$

If γ has a negative real Floquet multiplier, $\mu_{CZ}(\gamma)$ is odd. Otherwise, γ has a positive real Floquet multiplier and $\mu_{CZ}(\gamma)$ is even.

Example 3.4. Let γ be a simple closed Reeb orbit on a contact manifold of dimension 5 and suppose that all γ^k 's are nondegenerate. If γ is elliptic, either $\mu_{CZ}(\gamma) \in 2\mathbb{Z}$ and

$$\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 2) + 2[k\theta_1] + 2[k\theta_2] + 2, \quad \theta_1, \theta_2 \in (0, 1) \setminus \mathbb{Q}$$

or

$$\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).$$

If γ is hyperbolic,

$$\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).$$

If γ is neither elliptic nor hyperbolic, i.e. $e(\gamma) = 2$, then

$$\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 1) + 2[k\theta] + 1, \quad \theta \in (0, 1) \setminus \mathbb{Q}.$$

One can see that the Conley Zehnder index cannot decrease (resp. increase) under iteration if Σ is 3-dimensional and $\mu_{CZ}(\gamma)$ is positive (resp. negative). Unfortunately this does not remain true for higher dimensions. However if the Conley-Zehnder index of a simple closed Reeb orbit is big or small enough, we still have that property. We recall that our contact manifold (Σ, α) is of dimension $2n - 1$.

Proposition 3.5. *Let γ be a simple closed Reeb orbit with $\mu_{CZ}(\gamma) \geq n - 1$. Then we have*

$$\mu_{CZ}(\gamma^k) \leq \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.$$

PROOF. According to Theorem 3.2, the Conley-Zehnder index of the k -fold cover of γ is of the following form.

$$\mu_{CZ}(\gamma^k) = kr + \sum_{i=1}^j 2[k\theta_i] + j, \quad r + j = \mu_{CZ}(\gamma) \geq n - 1.$$

Since $j \in \{0, \dots, n - 1\}$ and $\theta_i \in (0, 1) \setminus \mathbb{Q}$, $r \geq 0$ and thus the claim follows directly. \square

Proposition 3.6. *If a simple closed Reeb orbit γ has $\mu_{CZ}(\gamma) = n + 1$, we have*

$$\mu_{CZ}(\gamma^k) + 2 \leq \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.$$

Moreover there exists $k_0 \in \mathbb{N}$ such that

$$\mu_{CZ}(\gamma^{k_0}) + 2 < \mu_{CZ}(\gamma^{k_0+1}).$$

PROOF. The first inequality follows from that $r \geq 2$ in the following form again.

$$\mu_{CZ}(\gamma^k) = kr + \sum_{i=1}^j 2[k\theta_i] + j, \quad r + j = n + 1$$

where $1 \leq j \leq n - 1$. If $r \geq 3$, $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 3$ for all $k \in \mathbb{N}$. If $r = 2$, we pick $k_0 \in \mathbb{N}$ satisfying $[k_0\theta] = 1$ so that $\mu_{CZ}(\gamma^{k_0+1}) \geq \mu_{CZ}(\gamma^{k_0}) + 4$. \square

Proposition 3.7. *If a simple closed Reeb orbit γ has $\mu_{CZ}(\gamma) \leq -n$,*

$$\mu_{CZ}(\gamma^k) > \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.$$

PROOF. We note that the Conley-Zehnder index of γ^k is

$$\mu_{CZ}(\gamma^k) = rk + \sum_{i=1}^j 2[k\theta_i] + j, \quad j \in \{0, 1, \dots, n - 1\}$$

and $r + j = \mu_{CZ}(\gamma) \leq -n$. Thus $r \leq -n - j < -2j$ and the claim is proved. \square

The following lemmas and corollary will be used in proving Theorem C.

Lemma 3.8. If for $(\theta_1, \dots, \theta_j) \in (0, 1)^j$, $1, \theta_1, \dots, \theta_j$ are rationally independent, then the sequence $(k\theta_1 - [k\theta_1], \dots, k\theta_j - [k\theta_j])$ is dense in $(0, 1)^j$.

PROOF. See [BCE07, Lemma 9]. \square

Lemma 3.9. Suppose that $\theta_1, \dots, \theta_j \in (0, 1)$, $j \geq 2$ are irrational and $\sum_{i=1}^j \theta_i$ is rational. Then there exists at least two $i_1, i_2 \in \{1, \dots, j\}$ such that $1, \theta_{i_1}, \theta_{i_2}$ are rationally independent.

PROOF. Assume by contradiction that there exists $p_i, q_i \in \mathbb{Q}$ such that $\theta_i = p_i\theta_1 + q_i$ for all $i \in \{1, \dots, j\}$. Then we have

$$\sum_{i=1}^j \theta_i = \sum_{i=1}^j p_i\theta_1 + \sum_{i=1}^j q_i.$$

This contradict to that θ_1 is irrational and proves the corollary. \square

Corollary 3.10. *Under the assumption of Lemma 3.9, there exists $k_0 \in \mathbb{N}$ such that*

$$\sum_{i=1}^j [k_0\theta_i] + 2 \leq \sum_{i=1}^j [(k_0 + 1)\theta_i].$$

PROOF. According to Lemma 3.8 and Lemma 3.9, there exists $k_0 \in \mathbb{N}$ and $i_1, i_2 \in \{1, \dots, j\}$ such that

$$k_0\theta_{i_1} - [k_0\theta_{i_1}] < \theta_{i_1} \quad \text{and} \quad k_0\theta_{i_2} - [k_0\theta_{i_2}] < \theta_{i_2}.$$

Thus $[k_0\theta_{i_1}] + 1 = [(k_0 + 1)\theta_{i_1}]$, $[k_0\theta_{i_2}] + 1 = [(k_0 + 1)\theta_{i_2}]$ and the claim is proved. \square

4. PROOFS OF THE MAIN RESULTS

4.1. Displaceable case.

This subsection is concerned with a contact manifold (Σ, ξ) which admits an exact contact embedding into a symplectic manifold (W, ω) which is convex at infinity and $c_1(W)|_{\pi_2(W)} = 0$. We continue to assume that the corresponding contact form α is nondegenerate.

Theorem 4.1. *Suppose that Σ is displaceable in (W, ω) . Then the S^1 -equivariant symplectic homology of (W, ω) vanishes.*

The above vanishing theorem can be proved by applying big theorems. Due to [CF09, AF10], displaceability of Σ in W implies vanishing of the Rabinowitz Floer homology of (W, Σ) . Then using a long exact sequence involving Rabinowitz Floer homology and Symplectic (co)homology in [CFO10] and a unit in symplectic cohomology, [Rit10] proved that vanishing of Rabinowitz Floer homology implies vanishing of symplectic homology. Since there exists a spectral sequence converging to $SH^{S^1}(W)$ with second page given by

$$E_{i,j}^2 \cong SH_i(W) \otimes H_j(\mathbb{C}P^\infty; \mathbb{Q}),$$

see [Vit99, BO12b], $SH^{S^1}(W)$ vanishes as well provided that Σ is displaceable in W . We remark that the last argument can be replaced by a different spectral sequence [Sei08, Section 8b]. However recently a direct relation between leafwise intersections and vanishing of $SH(W)$ and $SH^{S^1}(W)$ was studied in [Kan13] (also in [CO08]) in the case that (W, ω) is the completion of Liouville domain, i.e. $(W, \omega = \widehat{W}, \widehat{\omega})$. Therefore we have a direct proof of the theorem in that case and leave the following question.

Question 4.2. *If Σ is displaceable in (W, ω) , so is in $(\widehat{W}, \widehat{\omega})$? or a counter example?*

Combining the above theorem with the Viterbo long exact sequence we obtain the following computation which agrees with the contact homology computation [Yau04] in the subcritical Weinstein case.

Proposition 4.3. *If Σ is displaceable in (W, ω) , we have*

$$SH_*^{S^1,+}(W) \cong \bigoplus_{i+j=*+n-1} H_i(W_0, \Sigma; \mathbb{Q}) \otimes H_j(\mathbb{C}P^\infty; \mathbb{Q}).$$

PROOF. The S^1 -equivariant version of the Viterbo long exact sequence is

$$\cdots \rightarrow H_{*+n}^{S^1}(W_0, \Sigma; \mathbb{Q}) \rightarrow SH_*^{S^1}(W) \rightarrow SH_*^{S^1,+}(W) \rightarrow H_{*+n-1}^{S^1}(W_0, \Sigma; \mathbb{Q}) \rightarrow \cdots.$$

According to the above theorem, $SH_*^{S^1,+}(W) \cong H_{*+n-1}^{S^1}(W_0, \Sigma; \mathbb{Q})$. Since the S^1 -action on (W_0, Σ) is trivial and

$$H_{*+n-1}^{S^1}(W_0, \Sigma; \mathbb{Q}) \cong \bigoplus_{i+j=*+n-1} H_i(W_0, \Sigma; \mathbb{Q}) \otimes H_j(\mathbb{C}P^\infty; \mathbb{Q}).$$

□

Corollary 4.4. *If (Σ, α) is displaceable in (W, ω) , there exists a contractible closed Reeb orbit γ such that $\mu_{CZ}(\gamma) = n + 1$.*

PROOF. This directly follows from Proposition 4.3. We would like to mention that this result is not new and has been proved in various ways. □

A direct consequence of Proposition 2.1 and Proposition 4.3 is:

Corollary 4.5. *Suppose that Σ is displaceable in (W, ω) . Then,*

$$\frac{\sum_{i=1}^n b_i(W_0, \Sigma; \mathbb{Q})}{2} = \chi_m(W) = \sum_{\gamma_g \in \mathfrak{G}_s} \frac{(-1)^{\mu_{CZ}(\gamma_g)}}{\Delta(\gamma_g)} + \sum_{\gamma_b \in \mathfrak{B}_s} \frac{(-1)^{\mu_{CZ}(\gamma_b)}}{2\Delta(\gamma_b)}.$$

Proof of Theorem A.

Case (i). Suppose that $H_{2\ell-1}(W_0, \Sigma; \mathbb{Q}) \neq 0$ for some $\ell \in \mathbb{N}$ and that γ in Corollary 4.4 is the only closed Reeb orbit on (Σ, α) . Let γ_0 be a primitive of γ . Applying Proposition 4.3, we have

$$\begin{cases} SH_{n+1}^{S^1,+}(W) \cong \bigoplus_{i=1}^n H_{2i}(W_0, \Sigma; \mathbb{Q}), \\ SH_{2\ell-n}^{S^1,+}(W) \cong \bigoplus_{i=1}^{\ell} H_{2i-1}(W_0, \Sigma; \mathbb{Q}). \end{cases}$$

and thus multiples of γ_0 represents nonzero homology classes in $SH_{n+1}^{S^1,+}(W)$ and $SH_{2\ell-n}^{S^1,+}(W)$. However the parity of $n+1$ and the parity of $2\ell-n$ are different. Assume that the parity of $\mu_{CZ}(\gamma_0)$ is different from the parity of $n+1$. The other case follows in the same manner. Then all multiple covers of γ_0 with Conley-Zehnder index $n+1$ are bad orbits and thus do not contribute to $SH_{n+1}^{S^1,+}(W)$. This contradiction implies the existence of a second orbit geometrically different from γ_0 .

Case (ii). Suppose that $H_{2\ell}(W_0, \Sigma; \mathbb{Q}) = 0$ for all $0 \leq \ell \leq n-2$ and that $H_{2m-1}(W_0, \Sigma; \mathbb{Q}) = 0$ for all $m \in \mathbb{N}$. Assume on the contrary that there exists precisely one simple closed Reeb orbit γ_0 as above. According to Proposition 4.3, we have

$$\begin{cases} SH_{n-1}^{S^1,+}(W) \cong H_{2n-2}(W_0, \Sigma; \mathbb{Q}), \\ SH_{n-1+2j}^{S^1,+}(W) \cong H_{2n}(W_0, \Sigma; \mathbb{Q}) \oplus H_{2n-2}(W_0, \Sigma; \mathbb{Q}), \quad j \in \mathbb{N}, \\ SH_*^{S^1,+}(W) \cong \{0\}, \quad * \in \mathbb{Z} \setminus \{n-3+2j \mid j \in \mathbb{N}\}. \end{cases} \quad (4.1)$$

Subcase 1. If $\mu_{CZ}(\gamma_0) \geq n+1$, μ_{CZ} nondecreases under iteration, see Proposition 3.5, and thus $\mu_{CZ}(\gamma_0) = n+1$. But even in this case, $\mu_{CZ}(\gamma_0^k) + 2 \leq \mu_{CZ}(\gamma_0^{k+1})$ for all $k \in \mathbb{N}$ and there exists $k_0 \in \mathbb{N}$, $\mu_{CZ}(\gamma_0^{k_0}) + 2 < \mu_{CZ}(\gamma_0^{k_0+1})$ due to Proposition 3.6. This implies that there exist $j \in \mathbb{N}$ such that multiple covers of γ_0 cannot generate $SH_{n-1+2j}^{S^1,+}(W)$. Thus γ_0 cannot be the only one closed Reeb orbit.

Subcase 2. If $\mu_{CZ}(\gamma_0) < n-1$ or $\mu_{CZ}(\gamma_0) = n$, there exists $k \in \mathbb{N}$ such that $\mu_{CZ}(\gamma_0^k) = \mu_{CZ}(\gamma_0) + 1$ or $\mu_{CZ}(\gamma_0^k) = \mu_{CZ}(\gamma_0) - 1$ because of (4.1). But then γ_0^k is a bad orbit which does not contribute to S^1 -equivariant symplectic homology, we need another geometrically distinct closed Reeb orbit in this case as well.

Subcase 3. We assume that $\mu_{CZ}(\gamma_0) = n-1$. Due to the index iteration formula,

$$\mu_{CZ}(\gamma_0^k) = kr + \sum_{i=1}^j 2[k\theta_i] + j, \quad r + j = n-1$$

for some $\theta_i \in (0, 1) \setminus \mathbb{Q}$. Since $0 \leq j \leq n-1$, $r \geq 0$. If $r = 0$,

$$\mu_{CZ}(\gamma_0^k) = \sum_{i=1}^{n-1} 2[k\theta_i] + n-1, \quad \gamma_0 \in \mathfrak{G}_s$$

and we have the following identity due to Corollary 4.5.

$$\sum_{i=1}^{n-1} 2\theta_i = \Delta(\gamma_0) = \frac{2}{1 + b_{2n-2}(W_0, \Sigma; \mathbb{Q})}. \quad (4.2)$$

However according to the computation (4.1),

$$\mu_{CZ}(\gamma_0^{b_{2n-2}(W_0, \Sigma; \mathbb{Q})+1}) = n + 1$$

since $\mu_{CZ}(\gamma_0^k) \leq \mu_{CZ}(\gamma_0^{k+1})$ for all $k \in \mathbb{N}$ as observed in Proposition 3.5. Therefore there exists $i \in \{1, \dots, n-1\}$ such that $[(1 + b_{2n-2}(W_0, \Sigma; \mathbb{Q}))\theta_i] = 1$. But this contradicts to (4.2) and the fact that $\theta_i \in (0, 1) \setminus \mathbb{Q}$ for all $1 \leq i \leq n-1$.

If $r \geq 1$, then there exists $k_0 \in \mathbb{N}$ such that

$$\mu_{CZ}(\gamma_0^{k_0+1}) \geq \mu_{CZ}(\gamma_0^{k_0}) + 3$$

which contradicts to (4.1). Hence there exists a closed Reeb orbit geometrically distinct from γ_0 and this completes the proof. \square

Before proving Corollary A, we refer to Example 3.3 for the index iteration formula in the 3-dimensional case.

Proof of Corollary A.

According to Proposition 4.3,

$$\begin{cases} SH_1^{S^1,+}(W) \cong H_2(W_0, \Sigma; \mathbb{Q}), \\ SH_{2k}^{S^1,+}(W) \cong H_3(W_0, \Sigma; \mathbb{Q}), \\ SH_{2k+1}^{S^1,+}(W) \cong H_2(W_0, \Sigma; \mathbb{Q}) \oplus H_4(W_0, \Sigma; \mathbb{Q}), \end{cases} \quad (4.3)$$

for all $k \in \mathbb{N}$. If we write $SH_2^{S^1,+}(W) = \mathbb{Q}\langle\gamma_1, \dots, \gamma_{b_3(W_0, \Sigma; \mathbb{Q})}\rangle$, all γ_i 's are simple. Indeed if γ_i is not simple, it has to be a double cover of a simple one of Conley-Zehnder index 1 and thus bad. Therefore,

$$\mu_{CZ}(\gamma_i^k) = 2k, \quad i \in \{1, \dots, b_3(W_0, \Sigma; \mathbb{Q})\}, \quad k \in \mathbb{N}.$$

Since $\dim SH_3^{S^1,+}(W) \geq 1$, there exists another closed Reeb orbit v with $\mu_{CZ}(v) = 3$. If v is simple, there exists another closed Reeb orbit due to Proposition 3.6 to satisfy (4.3). Suppose that v is a multiple cover of a simple one, say v_0 , and that there is no simple closed Reeb orbit except v_0 and γ_i 's. Then $\mu_{CZ}(v_0) = 1$. If v_0 is hyperbolic, i.e. $\mu_{CZ}(v_0^k) = k$, only odd multiple covers take into account. Then there exists another simple closed Reeb orbit since $\dim SH_3^{S^1,+}(W) = \dim SH_1^{S^1,+}(W) + 1 \geq 2$. Suppose that v_0 is elliptic, i.e. $\mu_{CZ}(v_0^k) = 2[k\theta] + 1$ for some $\theta \in (0, 1) \setminus \mathbb{Q}$. Since v_0 generates all odd degrees of $SH_*^{S^1,+}(W)$,

$$2k + 1 = \mu_{CZ}(v_0^{kb_2(W_0, \Sigma; \mathbb{Q})+1}) = 2[(kb_2(W_0, \Sigma; \mathbb{Q}) + 1)\theta] + 1, \quad k \in \mathbb{N}.$$

By dividing both sides by k and taking a limit $k \rightarrow \infty$, we obtain a contradiction

$$\theta = \frac{1}{b_2(W_0, \Sigma; \mathbb{Q})} \in \mathbb{Q}.$$

This completes the proof. \square

4.2. Prequantization bundles.

Let (Q, Ω) be a symplectic manifold with an integral symplectic form Ω , i.e. $[\Omega] \in H^2(Q; \mathbb{Z})$. Since the first Chern class classifies isomorphism classes of complex line bundles, we can find a principal S^1 -bundle $p : P \rightarrow Q$ with $c_1(P) = k[\Omega]$ for $k \in \mathbb{N}$. Such a *prequantization bundle* P carries a connection 1-form α_{BW} such that the curvature form of α_{BW} is $-2k\pi\Omega$, i.e. $-2k\pi p^*\Omega = d\alpha_{BW}$, see [BW58] or [Gei08, Chapter 7.2]. Therefore a prequantization bundle $(P, \xi := \ker \alpha_{BW})$ is a contact manifold and especially (P, α_{BW}) is periodic, i.e. all Reeb flows are periodic. Suppose that $c_1(Q) = c[\Omega]$ for some $c \in \mathbb{Z}$. Due to the Gysin sequence for $S^1 \hookrightarrow P \xrightarrow{p} Q$, $0 = p^*c_1(P) = kp^*[\Omega]$ and thus $c_1(\xi) = \pi^*c_1(Q) = cp^*[\Omega]$ is a torsion class. Hence the Maslov indices for homologically trivial Reeb orbits are well defined. We remark that the generalized Maslov index due to [RS93] is well defined although the Conley-Zehnder index is not since (P, α_{BW}) is Morse-Bott. These two indices agree in the nondegenerate case. Suppose furthermore that (Q, ω) is simply connected and that $[\omega]$ is a primitive element in $H^2(Q)$. We denote by γ a principal orbit in P . Then since $\pi_1(P) = \mathbb{Z}_k$, the k -fold cover of γ is contractible and its Maslov index equals to $2c$, i.e. $\mu(\gamma^k) = 2c$, see [EGH00, Bou02].

We learned the following remark and proposition from Otto van Koert.

Remark 4.6. ([vK12]) In this remark we construct some examples which meet requirements in Theorem B. Let (B, ω) be a simply connected integral symplectic manifold such that $[\omega] \in H^2(B; \mathbb{Z})$ is primitive and $c_1(B) = a[\omega]$ for some $a \in \mathbb{Z}$. Let Q_k be a symplectic Donaldson hypersurface in (B, ω) Poincaré dual to $k[\omega]$ for sufficiently large $k \in \mathbb{N}$, see [Don96] and [CDvK12, Section 6]. Then according to [Gir02, Proposition 11], $W := B - \nu_B(Q_k)$ is a compact Weinstein. Here $\nu_B(Q_k)$ is the normal disk bundle over Q_k in B with $c_1(\nu_B(Q_k)) = k[\omega|_{Q_k}]$. Therefore the prequantization bundle (P, α_{BW}) over Q_k with $c_1(P) = k[\omega|_{Q_k}]$ has a Weinstein filling $(W, \omega|_W)$. Now we show that this example meets assumptions in Theorem B.

$$(i) \quad c_1(W)|_{\pi_2(W)} = 0$$

since $c_1(W) = c_1(B)|_W = a[\omega|_W] = a[d\lambda]$ for some 1-form λ on a Weinstein manifold W .

$$(ii) \quad Q_k \text{ is simply connected}$$

by (the analogue of) the Lefschetz hyperplane theorem, see [Don96, Proposition 39].

$$(iii) \quad c_1(Q_k) = (a - k)[\omega|_{Q_k}]$$

due to $c_1(Q_k) = c_1(B) - c_1(\nu_B(Q_k))$. Moreover if $\dim W \geq 8$, we have

$$(iv) \quad \pi_1(W) \cong \pi_1(\partial W)$$

since $\pi_2(W, \partial W)$ and $\pi_1(W, \partial W)$ are trivial. Indeed, since W is Weinstein, the Liouville flow can be used to make null-homotopy of elements in $\pi_1(W, \partial W)$ and $\pi_2(W, \partial W)$.

Proposition 4.7. ([vK12]) *Let $(P, \xi = \ker \alpha_{BW})$ be a prequantization bundle over a simply connected integral symplectic manifold (Q, ω) of dimension $(2n - 2)$ such that $[\omega]$ is primitive and $c_1(P) = k[\omega]$ for $k \in \mathbb{N}$. Suppose that $c_1(Q) = c[\omega]$ for some $|c| > n - 1$ and that (P, ξ) admits an exact contact embedding $i : (P, \xi) \hookrightarrow (W, d\lambda)$ with $c_1(W)|_{\pi_2(W)} = 0$ which is π_1 -injective. Then*

$$SH_*^{S^1, +}(W) \cong \bigoplus_{N=1}^{\infty} H_{*-(2Nc-n+1)}(Q; \mathbb{Q}).$$

PROOF. We compute the symplectic homology for $(W, \omega, P, \alpha_{BW})$, i.e. $\lambda|_{i(P)} = \alpha_{BW}$, and the resulting homology is an invariant for (W, ω, P, ξ) . We note that since i is injective on

π_1 -level, Morse-Bott components are exactly kN -fold covered fibers which we denote by P_{kN} . As in the subsection 2.4, there exists a Morse-Bott spectral sequence with E^1 -page

$$E_{pq}^1 = \bigoplus_{\substack{\mu(\gamma^{kN})=p; \\ N \in \mathbb{N}}} H_q^{S^1}(P_{kN}; \mathbb{Q})$$

converging to $SH_*^{S^1,+}(W)$. We observe that

$$\mu(\gamma^{kN}) = \mu_{Maslov}(\gamma^{kN}) - \frac{1}{2} \dim Q = 2cN - (n-1).$$

Since P is a principal circle bundle and every contractible closed Reeb orbits is good, we have

$$H_*^{S^1}(P_{kN}; \mathbb{Q}) \cong H_*(Q \times B\mathbb{Z}_{kN}; \mathbb{Q}) \cong H_*(Q; \mathbb{Q}).$$

Since $2c \geq 2n$ and the height of the spectral sequence is $\dim Q = 2n-2$, the spectral sequence stabilizes at the E^1 -page, see Figure 4.1, and thus we conclude

$$SH_*^{S^1,+}(W, d\lambda) \cong \bigoplus_{N \in \mathbb{N}} H_{*-(2Nc-(n-1))}(Q; \mathbb{Q}).$$

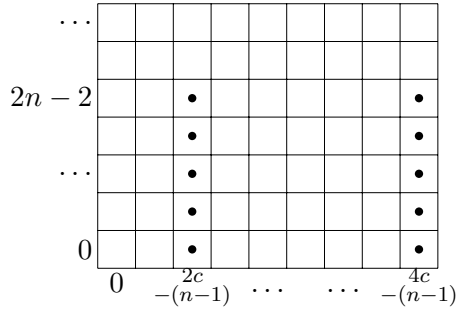


FIGURE 4.1. E^1 -page of Morse-Bott spectral sequence

□

Proof of Theorem B.

We observe that for all $N \in \mathbb{N}$,

$$SH_{2Nc-(n-1)}^{S^1,+}(W) = H_0(Q; \mathbb{Q}) = \mathbb{Q}, \quad SH_{2Nc+(n-1)}^{S^1,+}(W) = H_{2n-2}(Q; \mathbb{Q}) = \mathbb{Q}. \quad (4.4)$$

We first treat the case $c \geq n$. Note that $SH_*^{S^1,+}(W) = 0$ for all $*$ $< 2c - (n-1)$. Assume by contradiction that there is a precisely one simple closed Reeb orbit γ . If $\mu_{CZ}(\gamma)$ is smaller than $2c - (n-1)$, there has to be another closed Reeb orbit v of Conley-Zehnder index $\mu_{CZ}(\gamma) + 1$ or $\mu_{CZ}(\gamma) - 1$ since $SH_{\mu_{CZ}(\gamma)}^{S^1,+}(W) = 0$. But v cannot be a multiple cover of γ since otherwise v is a bad orbit. We may assume that $\mu_{CZ}(\gamma) \geq 2c - (n-1) \geq n+1$. Then since $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 2$ for all k , see Proposition 3.6, $\mu_{CZ}(\gamma)$ has to be $2c - (n-1)$ the first degree when $SH^{S^1,+}$ does not vanish.

We first exclude the case $\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma)$, $k \in \mathbb{N}$. Indeed, if it does, for some $k \in \mathbb{N}$,

$$k(2c - (n-1)) = \mu_{CZ}(\gamma^k) = 2c + (n-1).$$

If $k \geq 3$, $c \leq n - 1$. Let $k = 2$ and $c = (3n - 3)/2$. Moreover we know that for some $\ell \in \mathbb{N}$,

$$\ell(2n - 2) = \mu_{CZ}(\gamma^\ell) = 4c - (n - 1) = 5n - 5.$$

This contradiction proves the claim. Therefore the index iteration formula for γ has to be of the following form. Suppose that γ is good and the bad case is proved in the same way.

$$\mu_{CZ}(\gamma^k) = rk + \sum_{i=1}^j 2[k\theta_i] + j, \quad j \in \{1, \dots, n - 1\}$$

where $\theta_i \in (0, 1) \setminus \mathbb{Q}$ for all i . In particular we have

$$\mu_{CZ}(\gamma) = r + j = 2c - (n - 1)$$

Due to (4.4), there exist $k \in \mathbb{N}$ satisfying

$$\mu_{CZ}(\gamma^k) = 2c + n - 1.$$

Then since $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 2$ and $SH_*^{S^1,+}(W) = 0$ for all $2c + n - 1 < * < 4c - (n - 1)$ according to Proposition 4.7,

$$\mu_{CZ}(\gamma^{k+1}) = 4c - (n - 1).$$

Since $SH^{S^1,+}$ is periodic according to Proposition 4.7 again, we have for $N \in \mathbb{N}$,

$$\mu_{CZ}(\gamma^{(N-1)k+1}) = 2Nc - (n - 1), \quad \mu_{CZ}(\gamma^{Nk}) = 2Nc + (n - 1).$$

Since

$$2Nc = \mu_{CZ}(\gamma^{Nk+1}) - \mu_{CZ}(\gamma) = Nkr + \sum_{i=1}^j 2[(Nk + 1)\theta_i], \quad (4.5)$$

when $N = 1$, we obtain

$$r = \frac{2c - \sum_{i=1}^j 2[(k + 1)\theta_i]}{k}.$$

Again by (4.5), we have

$$N \sum_{i=1}^j [(k + 1)\theta_i] = \sum_{i=1}^j [(Nk + 1)\theta_i], \quad \text{for all } N \in \mathbb{N}.$$

But dividing out both sides by N and taking a limit $N \rightarrow \infty$, we deduce

$$\sum_{i=1}^j (k + 1)\theta_i = \sum_{i=1}^j k\theta_i$$

and this contradiction proves the theorem in the case $c \geq n$.

Now we consider the case $c \leq -n$. Due to Proposition 3.7, $\mu_{CZ}(\gamma^k) > \mu_{CZ}(\gamma^{k+1})$ and thus $\mu_{CZ}(\gamma) = 2c + n - 1$, see (4.4). As above we have for $N \in \mathbb{N}$,

$$\mu_{CZ}(\gamma^{(N-1)k+1}) = 2Nc + (n - 1), \quad \mu_{CZ}(\gamma^{Nk}) = 2Nc - (n - 1)$$

and this case is proved in a similar fashion. \square

4.3. Brieskorn spheres.

Let (Σ_a, ξ_a) be a Brieskorn sphere explained in the introductory section. The contact homologies of Brieskorn spheres were computed originally by [Ust99] and reproved using the Morse-Bott approach by [Bou02]. It is possible to compute the positive part of S^1 -equivariant symplectic homology of $V_\epsilon(a)$ a natural Weinstein filling of (Σ_a, ξ_a) in a similar way or using an isomorphism between those two homologies [BO12b]. Therefore we have

$$SH_*^{S^1,+}(V_\epsilon(a)) \begin{cases} 0 & * \in 2\mathbb{Z} + 1 \text{ or } * < n - 1, \\ \mathbb{Q} \oplus \mathbb{Q} & * \in 2\left[\frac{2N}{a_0}\right] + 2N(n-2) + n + 1, \ N \in \mathbb{N}, \ 2N + 1 \notin a_0\mathbb{Z}, \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

Proof of Theorem C.

Since $SH_*^{S^1,+}(V_\epsilon(a))$ does not vanish, there exists a simple closed Reeb orbit γ . Suppose that there is no another simple closed Reeb orbit except γ .

Case 1. If $\mu_{CZ}(\gamma) < n - 1$, there exists a closed Reeb orbit v with $\mu_{CZ}(v) = \mu_{CZ}(\gamma) + 1$ or $\mu_{CZ}(v) = \mu_{CZ}(\gamma) - 1$ since $SH_{\mu_{CZ}(\gamma)}^{S^1,+}(V_\epsilon(a)) = 0$. Since v has to be a good orbit, it cannot be a multiple cover of γ .

Case 2. If $\mu_{CZ}(\gamma) \geq n - 1$, $\mu_{CZ}(\gamma) = n - 1$ since $SH_{n-1}^{S^1,+}(V_\epsilon(a)) = \mathbb{Q}$ and $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k)$ due to Proposition 3.5. Thus the iteration formula for the Conley-Zehnder index of γ is

$$\mu_{CZ}(\gamma^k) = kr + \sum_{i=1}^j 2[k\theta_i] + j, \quad r + j = n - 1$$

where $j \in \{0, \dots, n - 1\}$. We claim that $r = 0$. If $r \in 2\mathbb{N}$, every multiple cover of γ is good and there exists $k_0 \in \mathbb{N}$ such that $\mu_{CZ}(\gamma^{k_0+1}) \geq \mu_{CZ}(\gamma^{k_0}) + 4$. This contradicts to that $\dim SH_*^{S^1,+}(V_\epsilon(a)) \geq 1$ for every even degree bigger than $n - 2$. If $r \in 2\mathbb{N} + 1$, only odd multiple covers are good and there exists $k_0 \in 2\mathbb{N} + 1$ such that $\mu_{CZ}(\gamma^{k_0+2}) \geq \mu_{CZ}(\gamma^{k_0}) + 4$. This is again a contradiction and proves the claim. Therefore a possible scenario is

$$\mu_{CZ}(\gamma^k) = \sum_{i=1}^{n-1} 2[k\theta_i] + n - 1.$$

We can easily compute the mean Euler characteristic

$$\chi_m(V_\epsilon(a)) = \lim_{N \rightarrow \infty} \frac{1}{N} \left(\frac{N}{2} + \frac{N}{4/p + 2(n-2)} - \frac{N}{4 + 2p(n-2)} \right) = \frac{1}{2} \left(\frac{1 + p(n-1)}{2 + p(n-2)} \right)$$

and due to the resonance identity, Proposition 2.1, we have

$$\sum_{i=1}^{n-1} \theta_i = \frac{2 + p(n-2)}{1 + p(n-1)}.$$

Since $n - 1 \geq 2$ from the construction of the Brieskorn spheres, we can apply Corollary 3.10. That is, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_{CZ}(\gamma^{k_0+1}) - \mu_{CZ}(\gamma^{k_0}) = \sum_{i=1}^{n-1} 2[(k_0 + 1)\theta_i] - \sum_{i=1}^{n-1} 2[k_0\theta_i] \geq 4$$

This contradicts to the computation that $SH_*^{S^1,+}(V_\epsilon(a))$ is nonzero for every even degree bigger than $n - 2$. \square

5. APPENDIX: MORE EXAMPLES

In this appendix we give examples which have two closed Reeb orbits even though they do not meet the requirements of Theorem A. For simplicity we treat 5-dimensional case, see Example 3.4.

Example 5.1. *Let (Σ, ξ) be a contact 5-manifolds which has displaceable exact contact embedding into (W, ω) which is convex infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. Suppose that a corresponding contact form is nondegenerate. If $b_2(W_0, \Sigma; \mathbb{Q}) = 1$ and $b_4(W_0, \Sigma; \mathbb{Q}) = 0$, there are two closed Reeb orbits contractible in W .*

PROOF. We may assume that $b_3(W_0, \Sigma; \mathbb{Q}) = b_5(W_0, \Sigma; \mathbb{Q}) = 0$ since otherwise the assertion is covered by Theorem A. According to Proposition 4.3, we have

$$SH_0^{S^1,+}(W) = SH_2^{S^1,+}(W) = \mathbb{Q}, \quad SH_{2k+2}^{S^1,+}(W) = \mathbb{Q} \oplus \mathbb{Q}, \quad k \in \mathbb{N}.$$

and

$$SH_*^{S^1,+}(W) = 0, \quad * \in \mathbb{Z} \setminus (2\mathbb{N} \cup \{0\}).$$

Suppose that there exists precisely one simple closed Reeb orbit γ . One can immediately see that γ cannot be hyperbolic, see Example 3.4. If $e(\gamma) = 2$, it has to be

$$\mu_{CZ}(\gamma^k) = -k + 2[k\theta] + 1, \quad k \in \mathbb{N}, \quad \theta \in (0, 1) \setminus \mathbb{Q}.$$

But Corollary 4.5 implies

$$1 = \frac{1}{2\Delta(\gamma)} = \frac{1}{2(2\theta - 1)}$$

which implies a contradiction $\theta = 3/4$. The remaining case is that γ is elliptic and

$$\mu_{CZ}(\gamma^k) = -2k + 2[k\theta_1] + 2[k\theta_2] + 2, \quad k \in \mathbb{N}, \quad \theta_i \in (0, 1) \setminus \mathbb{Q}.$$

Again by Corollary 4.5, we obtain

$$\theta_1 + \theta_2 = \frac{3}{2}.$$

Since both θ_1 and θ_2 are irrational, we have

$$[2k\theta_1] + [2k\theta_2] = [2k\theta_1] + [3k - 2k\theta_1] = 3k - 1, \quad k \in \mathbb{N}$$

and thus $\mu_{CZ}(\gamma^{2k}) = 2k$. From this we can derive $\mu_{CZ}(\gamma^{2k+1}) = 2k + 2$ for all $k \in \mathbb{N}$ since $|\mu_{CZ}(\gamma^{k+1}) - \mu_{CZ}(\gamma^k)| \leq 2$. This yields that

$$[(2k+1)\theta_1] + [(2k+1)\theta_2] = [(2k+1)\theta_1] + [3k+1+1/2 - (2k+1)\theta_1] = 3k+1,$$

and thus we have the following contradictory inequality.

$$(2k+1)\theta_1 - [(2k+1)\theta_1] < \frac{1}{2}, \quad k \in \mathbb{N}.$$

Indeed since $2\theta_1$ is irrational, there exists $k_0 \in \mathbb{N}$ such that $2k_0\theta_1 - [2k_0\theta_1] \approx 1 - \theta_1$. \square

Example 5.2. *Let (Σ, ξ) be a contact 5-manifolds which has displaceable exact contact embedding into (W, ω) which is convex infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. Suppose that a corresponding contact form is nondegenerate. If $b_2(W_0, \Sigma; \mathbb{Q}) = 1$ and $b_4(W_0, \Sigma; \mathbb{Q}) \geq 5$, there are two closed Reeb orbits contractible in W .*

PROOF. By the same reason as above we may assume that $b_3(W_0, \Sigma; \mathbb{Q}) = b_5(W_0, \Sigma; \mathbb{Q}) = 0$. According to Proposition 4.3, we have $SH_0^{S^1, +}(W) = \mathbb{Q}$ and $SH_2^{S^1, +}(W) = \mathbb{Q}^{\oplus(b_4(W_0, \Sigma; \mathbb{Q})+1)}$. Assume by contradiction that there exists precisely one simple closed Reeb orbit γ . The case that γ is not elliptic can be easily excluded. The only possible nontrivial case is that $\mu_{CZ}(\gamma) = 0$ and

$$\mu_{CZ}(\gamma^k) = -2k + 2[k\theta_1] + 2[k\theta_2] + 2, \quad \theta_i \in (0, 1) \setminus \mathbb{Q}, \quad k \in \mathbb{N}.$$

Corollary 4.5 yields that

$$\theta_1 + \theta_2 = \frac{b_4(W_0, \Sigma; \mathbb{Q}) + 2}{b_4(W_0, \Sigma; \mathbb{Q}) + 1} \leq \frac{7}{6}.$$

We note that $\mu_{CZ}(\gamma^k) \geq 2$ for $k \geq 2$ because of $SH_0^{S^1, +}(W) = \mathbb{Q}$ and $SH_*^{S^1, +}(W) = 0$ for all $* < 0$. Since $\mu_{CZ}(\gamma^2) \geq 2$, both θ_1 and θ_2 are bigger than $1/2$. In addition $\mu_{CZ}(\gamma^3) \geq 2$ implies that one of θ_1 and θ_2 is bigger than $2/3$. Thus we deduce

$$\theta_1 + \theta_2 > \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

This contradiction completes the proof. \square

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